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Spectral decomposition of the Renyi map

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Abstract. We construct a generalized spectral decomposition of the Frobenius–Perron operator of the general β-adic Renyi map using a general iterative operator method applicable in principle to any mixing dynamical system. We also explicitly define appropriate rigged Hilbert spaces, which provide mathematical meaning to the formally obtained spectral decomposition. The explicit construction of the eigenvalues and eigenvectors allows us to show that the essential spectral radius of the Frobenius–Perron operator decreases as the smoothness of the domain increases. The reason for the change of the spectrum from the unit disk to isolated eigenvalues, is the existence of coherent states of the Frobenius–Perron operator, which are not infinitely differentiable.

1. Introduction

Complex spectral decompositions for large Poincaré non-integrable dynamical systems have been recently constructed by the Brussels–Austin groups [1, 2]. The resonances appear as complex eigenvalues of the evolution generator. The solution algorithm of the eigenvalue problem defines an extension of the operator beyond the Hilbert space. The extension acquires meaning in a suitable [3] rigged Hilbert space [4].

Eigenfunctions of the Frobenius–Perron operator [5] of the baker [6] and the dyadic Renyi map [7] have also been recently obtained by Hasegawa and Saphir [6]. Gaspard [8] also constructed eigenfunctions for the general Renyi map through Euler's summation formula. The corresponding eigenvalues are just the Pollicott–Ruelle resonances [9–18]. The Pollicott–Ruelle resonances of a dynamical system are the complex poles of the meromorphic extension of the power spectrum (the Fourier transform of the correlation function). It is straightforward to see [19] that these poles are the logarithms of complex eigenvalues of the Frobenius–Perron operator of the dynamical system. The existence of Pollicott–Ruelle resonances has so far been shown only for axiom-A systems and the actual construction depends upon the possibility to construct Markov partitions [12]. For non-axiom-A systems, 'numerical studies are often the only accessible tool of investigation' [20]. The resonances of 'chaotic' systems may also be calculated if the unstable periodic orbits are known [21–25]. In any case there is no systematic method to construct the associated eigenfunctions and thus obtain a spectral decomposition of the Frobenius–Perron operator. Such a decomposition effectively answers directly all questions [11] concerning the decay properties (exponential or not) of the correlation functions, as well as the analyticity

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properties of the power spectra. Furthermore, the spectral decomposition of the Frobenius–Perron operator amounts to an effective solution of the prediction problem in terms of densities within the test function space.

In this paper, we construct a generalized spectral decomposition of the Frobenius–Perron operator of the β-adic Renyi map using a general algorithmic method [26]. This is only one example illustrating the proposed method which we expect to be applicable to any mixing dynamical system, as it is based on iterative solutions of operator equations. The origin of the method is the subdynamics decomposition of large Poincaré non-integrable dynamical systems [1, 2, 27–29]. We also explicitly define (section 4) suitable rigged Hilbert space structures, which provide a mathematical meaning to the formally obtained spectral decomposition. Non-polynomial eigenfunctions of the Frobenius–Perron operator are discussed in section 5. These eigenfunctions may also be seen as coherent states of the Frobenius–Perron operator. In section 6, we show how the spectral properties of the Frobenius–Perron operator of the β-adic Renyi maps depend upon the smoothness of the domain. The spectrum of the Frobenius–Perron operator changes from the whole unit disk to the isolated eigenvalues present in the spectral decomposition because the essential spectral radius of the Frobenius–Perron operator decreases as the smoothness of the domain increases.

The β-adic Renyi map S on the interval $[0, 1)$ is the multiplication, modulo 1, by the integer $\beta \geq 2$:

$$S : [0,1) \to [0,1) \ : \qquad x \mapsto Sx = \beta x \pmod 1 \, . \tag{1}$$

The Renyi maps arise in the β-adic digital representation of any number $x \in [0, 1)$.

The forward iterations of the Renyi map define a cascade which preserves the Lebesgue measure of $[0, 1)$. Renyi showed [30] that the Lebesgue measure is the only invariant measure, which means that the dynamical system is ergodic. Moreover, Rokhlin showed that the Renyi systems are highly unstable exact dynamical systems [31, 5], with positive Kolmogorov–Sinai entropy. Nowadays Renyi maps are considered to be the simplest models for 'chaos'.

The probability densities $\rho(x)$ evolve according to the Frobenius–Perron operator U [5]:

$$U\rho(x) \equiv \sum_{y, S(y)=x} \frac{1}{|S'(y)|}\rho(y) = \frac{1}{\beta}\sum_{r=0}^{\beta-1} \rho\left(\frac{x+r}{\beta}\right) . \tag{2}$$

The Frobenius–Perron operator is partial isometry on the Hilbert space L^2 of square-integrable densities over the unit interval. The Frobenius–Perron operator is the adjoint of the isometric Koopman operator $U^\dagger$:

$$U^\dagger \rho(x) = \rho(Sx) \, . \tag{3}$$

2. The spectral decomposition algorithm

The systematic method we use to obtain spectral decompositions of linear operators is a continuation of previous work done by the Brussels–Austin groups. The key idea is the construction of an intermediate operator Θ which is intertwined with the operator U:

$$U\Omega = \Omega\Theta \qquad \text{or} \qquad U = \Omega\Theta\Omega^{-1}. \tag{4}$$

The intertwining relation (4) was obtained by Prigogine *et al* [27] and George [28]. Recently, Petrosky and Prigogine [2] pointed out that the intertwining relation can be used for the construction of the spectral decomposition of the Liouville operator. The method as reformulated by us [26] may also be considered as a generalization of the partitioning technique of matrices and of the intertwining wave operator method of scattering theory.

The intermediate operator Θ is decomposable with respect to a complete family of projectors P_ν constructed from a suitably chosen biorthonormal system $|\varphi_n\rangle$, $\langle\widetilde{\varphi}_n|$. In the decomposition of the operator U into the diagonal U_0 and the non-diagonal part U_1,

$$\begin{aligned} U_0 &= \sum_n \langle\widetilde{\varphi}_n|U|\varphi_n\rangle|\varphi_n\rangle\langle\widetilde{\varphi}_n| = \sum_\nu \omega_\nu P_\nu \\ U_1 &= \sum_{m\neq n} \langle\widetilde{\varphi}_m|U|\varphi_n\rangle|\varphi_m\rangle\langle\widetilde{\varphi}_n| \end{aligned} \tag{5}$$

where ω_ν labels the identical diagonal elements $\langle\widetilde{\varphi}_n|U|\varphi_n\rangle$ (degenerate eigenvalues) of U_0 and P_ν is the ω_ν-eigenprojector ($\nu = n$ if there is no degeneracy). After constructing the spectral decomposition of Θ, the intertwining operator Ω provides the spectral decomposition of the operator U.

The intermediate operator Θ as well as the similarity operator Ω are obtained [26] from the creation and destruction operators C_ν and D_ν:

$$\Theta \equiv \sum_\nu (P_\nu U P_\nu + P_\nu U C_\nu P_\nu) \tag{6}$$

$$\Omega \equiv \sum_\nu (P_\nu + C_\nu) \tag{7a}$$

$$\Omega^{-1} \equiv \sum_\nu (P_\nu + D_\nu C_\nu)^{-1}(P_\nu + D_\nu). \tag{7b}$$

The creation and destruction operators C_ν and D_ν are constructed iteratively as solutions of the nonlinear equations for the components $P_\mu C_\nu$ and $D_\nu P_\mu$, ($\nu \neq \mu$):

$$[U_0, P_\mu C_\nu]_- = (P_\mu C_\nu - P_\mu)U_1(P_\nu + C_\nu) \tag{8a}$$

$$[U_0, D_\nu P_\mu]_- = (P_\nu + D_\nu)U_1(P_\mu - D_\nu P_\mu) \tag{8b}$$

or

$$(\omega_\mu - \omega_\nu)P_\mu C_\nu = (P_\mu C_\nu - P_\mu)U_1(P_\nu + C_\nu) \tag{9a}$$

$$(\omega_\nu - \omega_\mu)D_\nu P_\mu = (P_\nu + D_\nu)U_1(P_\mu - D_\nu P_\mu). \tag{9b}$$

If there is resonance we use the time-ordering boundary condition or regularization rule [26]. However, this case does not arise here, so the operators C_ν and D_ν are obtained from the equations

$$P_\mu C_\nu = \frac{1}{\omega_\mu - \omega_\nu}(P_\mu C_\nu - P_\mu)U_1(P_\nu + C_\nu) \tag{10a}$$

$$D_\nu P_\mu = \frac{1}{\omega_\nu - \omega_\mu}(P_\nu + D_\nu)U_1(P_\mu - D_\nu P_\mu) \,. \tag{10b}$$

Let us remark that (8)–(10) are a nonlinear generalization of the Lippmann–Schwinger equations for the Möller wave operators of scattering and may provide non-unitary intertwining operators even if the scattering asymptotic condition [32] between U_0 and U fails.

The algorithm for the construction of spectral decomposition of the operator U is the following:

(i) Choose a convenient biorthonormal system $|\varphi_n\rangle$, $\langle\tilde{\varphi}_n|$ and decompose the operator U into the diagonal part U_0 and the non-diagonal part or perturbation U_1 (equation (5)).

(ii) Construct the creation and destruction operators C_ν and D_ν iteratively as solutions of the equations (10), starting with $C_\nu^{[0]} = D_\nu^{[0]} = 0$.

(iii) Construct the intermediate operator Θ from (6) and find the spectral decomposition of Θ by solving the eigenvalue problem in each P_ν subspace. As Θ is not Hermitian, we expect that Θ may have a generalized Jordan decomposition into a diagonal part and a weighted shift [26].

(iv) Obtain the spectral decomposition of U from the spectral decomposition of Θ using the similarity Ω (equations (4) and (7)).

3. Spectral decomposition of the Frobenius–Perron operator of the Renyi map

We shall construct a spectral decomposition of the Frobenius–Perron operator (2) following the algorithm of the previous section.

(i) As the monomials x^n are eigenvectors of the dilatation operator $Vf(x) = f(\beta x)$, we choose

$$|\varphi_n\rangle = x^n \qquad \text{and} \qquad \langle\tilde{\varphi}_n| = \frac{(-1)^n \delta^{(n)}(x)}{n!} \tag{11}$$

as the initial biorthonormal system. This system gives the Taylor expansion of analytic functions

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \,.$$

The biorthonormality is straightforward:

$$\langle\tilde{\varphi}_m|\varphi_n\rangle = \int_0^1 \mathrm{d}x \frac{(-1)^m \delta^{(m)}(x)}{m!} x^n = \delta_{nm} \,. \tag{12}$$

The matrix elements of the Frobenius–Perron operator with respect to the biorthonormal system $|\varphi_n\rangle$ and $\langle\widetilde{\varphi}_n|$ are

$$U_{mn} = \langle\widetilde{\varphi}_m|U|\varphi_n\rangle = \begin{cases} \dfrac{1}{\beta^n} & \text{for } m=n \\ 0 & \text{for } m>n \\ \dfrac{1}{\beta^{n+1}} \dfrac{n(n-1)\cdots(n-m+1)}{m!} \displaystyle\sum_{k=1}^{\beta-1} k^{n-m} & \text{for } m<n\,. \end{cases}$$

Indeed,

$$U_{mn} = \int_0^1 \mathrm{d}x \frac{(-1)^m \delta^{(m)}(x)}{m!} \frac{1}{\beta} \sum_{k=0}^{\beta-1} \left(\frac{x+k}{\beta}\right)^n = \frac{1}{m!} \frac{1}{\beta^{n+1}} \sum_{k=0}^{\beta-1} \left[\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x+k)^n\right]_{x=0}.$$

The Frobenius–Perron operator is represented, therefore, as an upper triangular matrix with respect to the biorthonormal system $|\varphi_n\rangle$ and $\langle\widetilde{\varphi}_n|$:

$$U_0 = \sum_{n=0}^{\infty} \frac{1}{\beta^n} |\varphi_n\rangle\langle\widetilde{\varphi}_n| \equiv \sum_{n=0}^{\infty} \frac{1}{\beta^n} P_n \qquad U_1 = \sum_{m<n} U_{mn} |\varphi_m\rangle\langle\widetilde{\varphi}_n| \tag{13}$$

with $P_n = |\varphi_n\rangle\langle\widetilde{\varphi}_n| = P_\nu$ as there is no degeneracy in U_0.

(ii) It is a general property [26] of (upper) triangular decompositions that the corresponding creation and destruction operators C_ν and D_ν are also upper triangular operators. Furthermore, the nonlinear operator equations (10) for C_ν and D_ν become linear and they have the form of Lippmann–Schwinger equations of scattering [32]:

$$P_m C_n = \frac{1}{\omega_n - \omega_m} P_m U_1 (P_n + C_n) \tag{14a}$$

$$D_n P_m = \frac{1}{\omega_n - \omega_m} (P_n + D_n) U_1 P_m\,. \tag{14b}$$

In our case, of course, $\omega_n = 1/\beta^n$, $n = 0, 1, 2, \ldots$. The triangular form of C_n and D_n follows from (10) by induction. For the operator C_n

$$P_m C_n^{[1]} = -\frac{1}{\omega_m - \omega_n} P_m U_1 P_n$$

is upper triangular without diagonal parts because U_1 is also upper triangular. Suppose that $P_m C_n^{[k]}$, $k \geq 1$ is upper triangular without diagonal parts. Then

$$P_m C_n^{[k+1]} = \frac{1}{\omega_m - \omega_n} (P_m C_n^{[k]} - P_m) U_1 (P_n + C_n^{[k]})$$

is also upper triangular without diagonal parts because it is the product of three upper triangular operators without diagonal parts.

The proof for the operator D_n is the same.

The simplification (14) follows immediately from the lemma

$$P_n U_1 (P_n + C_n) = (P_n + D_n) U_1 P_n = 0. \tag{15}$$

Lemma (15) follows from the triangular property of U_1 and C_n:

$$P_n U_1 (P_n + C_n) = P_n U_1 P_n + P_n U_1 C_n = 0 + \sum_{n'} P_n L_1 P_{n'} C_n = 0$$

because $P_n U_1 P_{n'} = 0$ for $n \geq n'$ and $P_{n'} C_n = 0$ for $n' \geq n$.

The proof for D_n is the same.

From (14) we obtain (see appendix) the matrix elements of C_n and D_n:

$$C_{mn} \equiv \langle \tilde{\varphi}_m | C_n | \varphi_n \rangle = \begin{cases} \dfrac{n! B_{n-m}}{m!(n-m)!} & \text{for } m < n \\ 0 & \text{for } m \geq n \end{cases} \tag{16a}$$

$$D_{nm} \equiv \langle \tilde{\varphi}_n | D_n | \varphi_m \rangle = \begin{cases} \dfrac{m!}{n!(m-n+1)!} & \text{for } m > n \\ 0 & \text{for } m \leq n\,. \end{cases} \tag{16b}$$

B_{m-n} are the Bernoulli numbers, defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\,. \tag{17}$$

(iii) For triangular operators the intermediate operator Θ is identical with the diagonal part of U, U_0 [26], namely

$$\Theta = U_0 = \sum_{n=0}^{\infty} \frac{1}{\beta^n} |\varphi_n\rangle\langle\tilde{\varphi}_n|\,. \tag{18}$$

This follows straightforwardly from (6) and the lemma (15).

(iv) For triangular operators, the similarity operators Ω and Ω^{-1} have also a simple form [26]:

$$\Omega = \sum_n (P_n + C_n) \tag{19a}$$

$$\Omega^{-1} = \sum_n (P_n + D_n)\,. \tag{19b}$$

These formulae follow from (7) and the lemma

$$D_n\ C_n = 0\,. \tag{20}$$

This lemma follows from triangular property of C_n and D_n

$$D_n C_n = \sum_{n'} P_n D_n P_{n'} P_{n'} C_n P_n = 0$$

because $P_n D_n P_{n'} = 0$ for $n \geq n'$ and $P_{n'} C_n P_n = 0$ for $n' \geq n$.

The right and left eigenvectors of U are obtained from the eigenvectors $|\varphi_n\rangle$, $\langle\widetilde{\varphi}_n|$ of Θ and the operators Ω, Ω^{-1}. The right eigenvectors of U are

$$\Omega|\varphi_n\rangle = |\varphi_n\rangle + \sum_m C_{mn}|\varphi_m\rangle = \left|x^n + \sum_{m=0}^{n-1} x^m \frac{n!}{m!(n-m)!} B_{n-m}\right\rangle = |B_n(x)\rangle\,.$$

$B_n(x)$ is the n-degree Bernoulli polynomial defined by the generating function [33, §9]

$$\frac{z\mathrm{e}^{zx}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n\,. \tag{21}$$

Formula (21) gives the relation between the Bernoulli polynomials and Bernoulli numbers: $B_n(0) = B_n$.

The left eigenvectors of U are

$$\begin{aligned}
\langle\widetilde{\varphi}_n|\Omega^{-1} &= \langle\widetilde{\varphi}_n| + \sum_m D_{nm}\langle\widetilde{\varphi}_m| = \left\langle \frac{(-1)^n\delta^{(n)}(x)}{n!} + \sum_{m=n+1}^{\infty} \frac{(-1)^m\delta^{(m)}(x)}{n!(m-n+1)!}\right| \\
&= \left\langle \frac{1}{n!}\sum_{l=0}^{\infty} \frac{(-1)^{l+n}\delta^{(l+n)}(x)}{(l+1)!}\right| = \left\langle \frac{(-1)^n}{n!}\int_0^1 \mathrm{d}x' \sum_{l=0}^{\infty} \frac{(-1)^l\delta^{(l+n)}(x)}{l!} x'^l\right| \\
&= \left\langle \frac{(-1)^n}{n!}\int_0^1 \mathrm{d}x'\,\delta^{(n)}(x-x')\right| \\
&= \begin{cases} \langle 1| & n = 0 \\ \left\langle \dfrac{(-1)^{(n-1)}}{n!}\{\delta^{(n-1)}(x-1) - \delta^{(n-1)}(x)\}\right| & n = 1, 2, \ldots \end{cases} \\
&\equiv \langle\widetilde{B}_n|\,.
\end{aligned}$$

The left eigenvectors $\langle\tilde{B}_n|$ are meaningful as antilinear functionals acting on analytic test functions only.

The spectral decomposition of U is therefore

$$\begin{aligned}
U &= \sum_{n=0}^{\infty} \frac{1}{\beta^n}\Omega|\varphi_n\rangle\langle\widetilde{\varphi}_n|\Omega^{-1} = \sum_{n=0}^{\infty} \frac{1}{\beta^n}|B_n\rangle\langle\tilde{B}_n| \\
U\rho(x) &= \int_0^1 \mathrm{d}x'\rho(x') + \sum_{n=1}^{\infty} \frac{\rho^{(n-1)}(1) - \rho^{(n-1)}(0)}{n!\beta^n} B_n(x)\,.
\end{aligned} \tag{22}$$

The orthonormality of the system $|B_n\rangle$ and $\langle\tilde{B}_n|$ follows immediately, while the completeness relation is just the Euler–MacLaurin summation formula for the Bernoulli polynomials [33, §9]

$$\rho(x) = \int_0^1 \mathrm{d}x'\rho(x') + \sum_{n=1}^{\infty} \frac{\rho^{(n-1)}(1) - \rho^{(n-1)}(0)}{n!} B_n(x)\,. \tag{23}$$

The fact that the Bernoulli polynomials are eigenfunctions of the Frobenius–Perron operator of the Renyi map also follows immediately from the well known but not frequently quoted formula [34, p 284, as an exercise]

$$B_n(\beta x) = \beta^{n-1} \sum_{r=0}^{\beta-1} B_n\left(x + \frac{r}{\beta}\right). \tag{24}$$

This fact was 'rediscovered' by Hasegawa and Saphir [6, 7] and by Gaspard [8]. They also point out that the eigenvalues corresponding to the Bernoulli polynomials coincide with the Pollicott–Ruelle resonances.

The Bernoulli polynomials are the only polynomial eigenfunctions as any polynomial can be uniquely expressed as a linear combination of the Bernoulli polynomials.

4. The meaning of the spectral decomposition of the Frobenius–Perron operator

The spectral decomposition (22) of U, constructed previously, has no meaning in the Hilbert space L^2 as the derivatives $\delta^{(n)}(x)$ of Dirac's delta function appear as left eigenvectors of U. The left eigenvectors [35, p 29] of an operator U are eigenvectors of its adjoint $U^\dagger$ corresponding to the complex conjugate eigenvalue

$$\begin{aligned} \langle \tilde{f} | U &= z \langle \tilde{f} | \\ \langle U^\dagger \tilde{f} | &= \langle z^* \tilde{f} | \end{aligned} \tag{25}$$

and

$$U^\dagger | \tilde{f} \rangle = z^* | \tilde{f} \rangle . \tag{26}$$

Following the standard convention of Dirac [35], $|\ \rangle$ denotes linear functionals, while $\langle\ |$ denotes antilinear functionals:

$$\langle g | c_1\phi_1 + c_2\phi_2 \rangle = c_1 \langle g | \phi_1 \rangle + c_2 \langle g | \phi_2 \rangle \qquad \langle c_1\phi_1 + c_2\phi_2 | g \rangle = c_1^* \langle \phi_1 | g \rangle + c_2^* \langle \phi_2 | g \rangle$$

Furthermore, as the Renyi transformations are exact [31, 5] dynamical systems, the Koopman operator $U^\dagger$ is a unilateral shift of infinite multiplicity. Unilateral shifts [36, 37], called semi-unitary operators by Rokhlin [31], do not admit a spectral decomposition in Hilbert space, as they cannot be decomposed in any way at all. In fact, the undecomposability is a necessary and sufficient condition for an isometry to be a unilateral shift [37, ch 7, p 110].

A natural way to give meaning to formal eigenvectors of operators which do not admit eigenvectors in Hilbert space, like the Koopman operator $U^\dagger$, is to extend the operator to a suitable rigged Hilbert space, e.g. $\Phi \subset L^2 \subset \Phi^\dagger$ [38, ch I]. $\Phi^\dagger$ is the space of continuous linear functionals on the properly chosen test function space Φ. In this way, the spectral decomposition (22) can be understood as a natural generalization of the Gelfand [38, ch I, 39, ch IV]–Maurin [40] theory of generalized spectral decompositions of self-adjoint operators with continuous spectrum.

The extension of $U^\dagger$ to $\Phi^\dagger$ is defined in the standard way [38, I.4]:

$$\langle U^\dagger f|\phi\rangle = \langle f|U\phi\rangle \tag{27}$$

for all test functions ϕ in Φ.

It is therefore necessary that the test function space is stable with respect to the Frobenius–Perron operator U, namely $U\Phi \subset \Phi$.

A suitable test function space is the subspace $\mathcal{P}$ of polynomials. The space $\mathcal{P}$ is dense [41, p 21] in L^2 and a nuclear LF-space [42, ch 51] and thus, complete and barrelled. Moreover, the space $\mathcal{P}$ is stable with respect to the Frobenius–Perron operator U and U is continuous with respect to the topology of $\mathcal{P}$, because U preserves the degree of polynomials.

From the previous discussion the adjoint $U^\dagger$ can be continuously extended to the topological dual $\mathcal{P}^\dagger$. This is therefore an appropriate rigged Hilbert space to give meaning to the spectral decomposition of U.

One may of course look for a tighter rigging. The corresponding test functions ϕ should at least provide a domain for the Euler–MacLaurin summation formula (23). The requirement of absolutely convergence of the series (23) means that

$$\sum_{n=1}^{\infty}\left|\frac{\phi^{(n-1)}(y)}{n!}B_n(x)\right| < \infty \qquad (y = 0, 1)\,. \tag{28}$$

We shall show that the condition (28) implies that the test functions ϕ should be restrictions on $[0, 1)$ of entire functions of exponential type c with $0 < c < 2\pi$. The space $\mathcal{E}_c$ of entire functions of exponential type $c > 0$ consists of the entire functions $\phi(z)$ such that

$$|\phi(z)| \le K e^{c|z|} \qquad \forall z \in \mathbf{C} \quad \text{for some } K > 0\,.$$

Indeed, Cauchy's criterion for the convergence of the positive series gives

$$\limsup_{n\to\infty}\left[\frac{|B_n(x)|}{n!}|\phi^{(n-1)}(y)|\right]^{1/n} < 1 \qquad (y = 0, 1)\,.$$

Then, with the aid of the inequality

$$\limsup_{n\to\infty} a_n b_n \ge \liminf_{n\to\infty} a_n \limsup_{n\to\infty} b_n$$

for any positive sequences a_n, b_n and the property of the Bernoulli polynomials [33, §9(i)]

$$\liminf_{n\to\infty}\left[\frac{|B_n(x)|}{n!}\right]^{1/n} = \frac{1}{2\pi}$$

we obtain

$$\limsup_{n\to\infty}\left[|\phi^{(n-1)}(0)|\right]^{1/n} < 2\pi\,. \tag{29}$$

As shown by Boas [43, ch 2], condition (29) is necessary and sufficient for the function $\phi(z)$ to be an entire function of exponential type c with $0 < c < 2\pi$.

The fact that any entire function of exponential type less than 2π has a convergent Euler–MacLaurin expansion (23) is mentioned by Boas and Buck [33, §9]. It is clear that we have therefore the whole family $\mathcal{E}_c$, $0 < c < 2\pi$ of test function spaces. Each space $\mathcal{E}_c$ is a Banach space with norm [42, ch 22]

$$\|\phi\|_c \equiv \sup_{z\in\mathbf{C}} |\phi(z)| \mathrm{e}^{-c|z|} .$$

The norm topology of each $\mathcal{E}_c$ is stronger than the Hilbert space topology [42, ch 22]. Each space $\mathcal{E}_c$ is dense in the Hilbert space L^2 as $\mathcal{E}_c$ includes the polynomial space $\mathcal{P}$.

Each test space $\mathcal{E}_c$ is stable under the Frobenius–Perron operator U. Indeed, for every function $\phi(x)$ in $\mathcal{E}_c$

$$\begin{aligned} |U\phi(x)| &= \left|\frac{1}{\beta}\sum_{r=0}^{\beta} \phi\left(\frac{x+r}{\beta}\right)\right| \le \frac{1}{\beta}\sum_{r=0}^{\beta}\left|\phi\left(\frac{x+r}{\beta}\right)\right| \\ &\le \frac{K}{\beta}\sum_{r=0}^{\beta-1} \exp\left\{c\frac{|z+r|}{\beta}\right\} \le \left(\frac{K}{\beta}\sum_{r=0}^{\beta-1} \mathrm{e}^{rc/\beta}\right) \exp\left\{\frac{c}{\beta}|z|\right\} . \end{aligned}$$

The last inequality implies that $U\phi$ is of exponential type c/β. The stability of each $\mathcal{E}_c$ follows from the nesting property [42, ch 22] $\mathcal{E}_c \subset \mathcal{E}_{c'}$ if $c < c'$. The nuclear space $\mathcal{P}$ and the family of Banach spaces $\mathcal{E}_c$ form reasonable test function spaces providing increasingly tighter riggings, which give meaning to the spectral decomposition (22).

As discussed in the beginning of this section, the Koopman operator $U^\dagger$ is a unilateral shift. As unilateral shifts are not decomposable, they cannot have eigenfunctions in the Hilbert space. However, the extended Koopman operator has generalized eigenvectors

$$U^\dagger |\tilde{B}_n\rangle = \frac{1}{\beta^n} |\tilde{B}_n\rangle \tag{30}$$

and generalized spectral decomposition

$$U^\dagger = \sum_{n=0}^{\infty} \frac{1}{\beta^n} |\tilde{B}_n\rangle\langle B_n| . \tag{31}$$

This is, to our knowledge, the first example of non-normal operators which do not admit a spectral decomposition in the Hilbert space but possess a generalized spectral decomposition.

5. Non-polynomial eigenfunctions of the Frobenius–Perron operator

It is straightforward to see that the Frobenius–Perron operator U has also the non-polynomial eigenfunctions:

$$U\ \psi_z^s(x) = z\psi_z^s(x) \qquad \psi_z^s(x) \equiv \sum_{n=0}^{\infty} z^n g_n^s(x) \quad s \in \boldsymbol{Z}_r \tag{32}$$

where $Z_r \equiv \{s \in Z : s/\beta \notin Z\}$ and

$$g_n^s(x) \equiv \exp(2\pi i s \beta^n x) \qquad n = 0, 1, 2, \dots . \tag{33}$$

It is also straightforward to see that the functions $\{g_n^s(x)\}$ satisfy the property

$$U g_n^s(x) = \begin{cases} g_{n-1}^s(x) & (n \geq 1) \\ 0 & (n = 0) . \end{cases} \tag{34}$$

Equation (34) means that the operator U is the adjoint unilateral shift in the orthogonal complement of constant functions $L^2 \ominus \mathrm{C}$ and that the set $\{g_0^s(x)\}_{s\in Z_r}$ is a basis of the wandering generating subspace [36] of the shift. As the set $\{g_n^s(x)\}$ is essentially the Fourier basis of $L^2 \ominus C$, it forms an orthonormal basis of the space $L^2 \ominus C$.

The eigenfunctions $\psi_z^s(x)$ have the following properties:

(i) Although the family $\{\psi_z^s(x)\}$ is an overcomplete nonorthonormal set in $L^2 \ominus C$, for any eigenvalue z with $|z| < 1$, each subfamily $\{\psi_z^s(x)\}_{s\in Z_r}$ is an orthonormal basis for the z-eigenspace. In fact, if $\psi_z(x) = \sum_{s\in Z_r} \sum_{n=0}^{\infty} c_n^s \; g_n^s(x) \in L^2$ is an eigenfunction of U with eigenvalue z,

$$U \; \psi_z(x) = z \psi_z(x)$$

equation (34) gives

$$c_{n+1}^s = z c_n^s = \cdots = z^n c_0^s$$

and thus

$$\psi_z(x) = \sum_{s\in Z_r} \sum_{n=0}^{\infty} z^n \; c_0^s \; g_n^s(x) = \sum_{s\in Z_r} c_0^s \psi_z^s(x) .$$

(ii) The eigenfunctions $\psi_z^s(x)$ are m-times continuously differentiable on $(0, 1)$ for a non-negative integer m satisfying

$$\frac{-\ln|z|}{\ln\beta} > m \geq \frac{-\ln|z|}{\ln\beta} - 1 . \tag{35}$$

Indeed, the term-by-term νth derivative of (32) is

$$\sum_{n=0}^{\infty} z^n \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} g_n^s(x) = (2\pi i s)^\nu \sum_{n=0}^{\infty} (z\beta^\nu)^n \; g_n^s(x) . \tag{36}$$

Let $0 \leq \nu \leq m$; then because of $|g_n^s(x)| = 1$ and $|z|\beta^\nu \leq |z|\beta^m < 1$ by (35), the series (36) converges absolutely and uniformly on $[0, 1]$. Thus, $\psi_z^s(x)$ is m-times continuously differentiable and its νth derivative, $\nu = 1, 2, \dots m$, is given by (36).

(iii) Moreover, the mth derivative $\psi_z^{s(m)}(x)$ of the eigenfunction $\psi_z^s(x)$ is Hölder continuous of exponent α: $0 < \alpha < -\ln(|z\beta^m|)/\ln\beta$. The proof follows from the inequality

$$|\exp(ix) - 1| \leq 2^{1-\alpha}|x|^\alpha \qquad (x \in R \; 0 < \alpha < 1) . \tag{37}$$

We have

$$|g_n^s(x) - g_n^s(y)| = |\exp(2\pi \mathrm{i} s\beta^n(x-y)) - 1| \le 2(\pi|s|)^\alpha \beta^{\alpha n}|x-y|^\alpha .$$

Therefore, for $x \neq y$ and $|z|\beta^{m+\alpha} < 1$,

$$\frac{|\psi_z^{s(m)}(x) - \psi_z^{s(m)}(y)|}{|x-y|^\alpha} \le (2\pi|s|)^m \sum_{n=0}^{\infty}(|z|\beta^m)^n \frac{|g_n^s(x) - g_n^s(y)|}{|x-y|^\alpha}$$
$$\le 2^{m+1}(\pi|s|)^{m+\alpha} \sum_{n=0}^{\infty}[|z|\beta^{m+\alpha}]^n < +\infty .$$

(iv) One can express the Bernoulli polynomials $B_n(x)$ in terms of the eigenfunctions $\psi_z^s(x)$ by rearranging the Fourier expansions of the Bernoulli polynomials [34, p 284]:

$$B_0(x) = 1 \qquad \frac{B_n(x)}{n!} = \sum_{s\in Z_r} \frac{-1}{(2\pi \mathrm{i} s)^n} \psi^s_{1/\beta^n}(x) \quad (n = 1, 2, \ldots) . \tag{38}$$

(v) The eigenfunctions $\psi_z^s(x)$ for $1 > |z| \ge 1/\beta$ can be expressed in terms of the Weierstrass functions as follows:

$$\psi_z^s(x) = \sum_{n=0}^{\infty} z^n \cos(2\pi\beta^n sx) + \mathrm{i}\sum_{n=0}^{\infty} z^n \sin(2\pi\beta^n sx) . \tag{39}$$

Such functions are called complex Weierstrass functions by Mandelbrot [44, p 388]. The fact that the Weierstrass functions appear as eigenfunctions of the Frobenius–Perron operator has also been noticed by Hata [45]. A pictorial representation of these functions is presented in figures 1 and 2.

These curves are fractal sets. Indeed, by using the Besicovitch–Ursell inequality [46], it is easy to show that the Hausdorff dimension D_z of the graphs of $\mathrm{Re}\,\psi_z^s(x)$ and $\mathrm{Im}\,\psi_z^s(x)$ satisfies

$$1 \le D_z \le 2 + \frac{\ln|z|}{\ln\beta} . \tag{40}$$

(vi) Let us remark before closing this section that the behaviour of the eigenfunctions $\psi_z^s(x)$ of the Frobenius–Perron operator U resembles the behaviour of the coherent states. Coherent states are eigenfunctions of the annihilation operator [47], which is an unbounded adjoint weighted shift [36]. This is, in fact, a special case of a general construction of coherent states for shift operators [48].

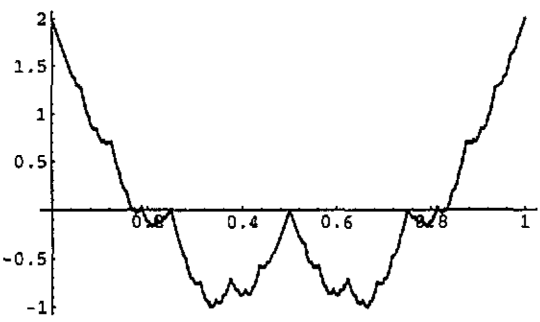

Figure 1. The graph of $y = \mathrm{Re}\,\psi^1_{1/2}(x)$ in the case of $\beta = 2$.

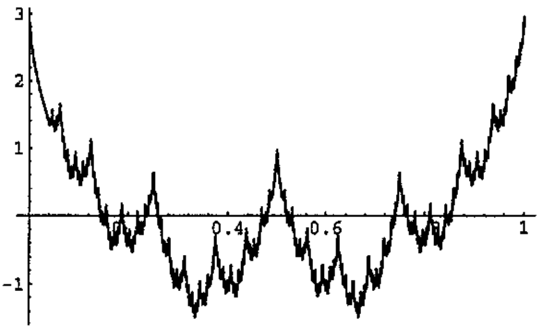

Figure 2. The graph of $y = \mathrm{Re}\,\psi^{1}_{2/3}(x)$ in the case of $\beta = 2$.

6. Domain dependence of the spectrum of the Frobenius–Perron operator

As the Frobenius–Perron operator U is the adjoint shift, the spectrum of U on the Hilbert space $L^2 \ominus \mathbf{C}$ is the unit disk [36, Problem 82]. How does the spectrum of U change to isolated eigenvalues in the domain $\mathcal{P}$ or $\mathcal{E}_c$, $0 < c < 2\pi$? In order to answer this question, we shall study the spectral radii of the restrictions of U in the intermediate dense domains $C^{m,\alpha}$ of all m-times continuously differentiable functions on $[0,1]$ whose mth derivative is Hölder-continuous with exponent α, $0 < \alpha \le 1$. It is obvious that $\mathcal{P} \subset \mathcal{E}_c \subset \mathcal{E}_{c'} \subset C^{m,\alpha} \subset C^{m',\alpha'} \subset L^2$ with $m + \alpha < m' + \alpha'$ and $c < c'$.

The difference of the Frobenius–Perron operator U from the spectral decomposition (22) is expressed by the operators

$$U_{[m]}\phi(x) \equiv U\phi(x) - \int_0^1 \mathrm{d}x'\phi(x') - \sum_{n=1}^{m} \frac{\phi^{(n-1)}(1) - \phi^{(n-1)}(0)}{n!\ \beta^n} B_n(x)$$
$$m = 1, 2, \ldots . \tag{41}$$

Each operator $U_{[m]}$ is well-defined on $C^{m,\alpha}$ $m = 1, 2, \ldots$ and one can show that:

(i) The range of $U_{[m]}$ is the subspace $C^{m,\alpha}_p$ of all m-times continuously differentiable functions on $[0,1]$ whose mth derivative is Hölder-continuous with exponent $0 < \alpha \le 1$ and all derivatives are periodic, i.e. $\phi^{(\nu)}(0) = \phi^{(\nu)}(1)$ $(\nu = 0, \ldots, m-1)$.

(ii) The operator $U_{[m]}$ has spectrum in the closed disk $|z| \le 1/\beta^{m+\alpha}$ with associated eigenfunctions the non-polynomial eigenfunctions $\psi^{s}_{z}(x)$ given by (32). This means that the spectral radius of the operator $U_{[m]}$ becomes smaller as the smoothness $(m + \alpha)$ increases. When $m \to \infty$, the spectral radius approaches zero. This implies that, as $m \to \infty$, the Frobenius–Perron operator U is 'effectively' approximated by the spectral decomposition (22) which is associated with isolated eigenvalues and analytic functions.

Before going to the proof, we remind the reader that the spaces $C^{m,\alpha}$ and $C^{m,\alpha}_p$ are Banach spaces with norms

$$\|\phi\|_{m,\alpha} \equiv \sum_{j=0}^{m} \|\phi^{(j)}\|_\infty + \|\phi^{(m)}\|_\alpha \tag{42}$$

where $\|\phi^{(j)}\|_\infty = \sup_{x\in[0,1]} |\phi^{(j)}(x)|$ the supremum norm and

$$\|\phi^{(m)}\|_\alpha \equiv \sup\left\{ \frac{|\phi^{(m)}(x) - \phi^{(m)}(y)|}{|x-y|^\alpha} : x, y \in [0,1], x \neq y \right\}. \tag{43}$$

the minimum α-Hölder constant.

Proof of (i) and (ii). The periodicity condition $(U_{[m]}\phi)^{(\nu)}(1) = (U_{[m]}\phi)^{(\nu)}(0)$ $(\nu = 0, \ldots, m-1)$ for (i) follows straightforwardly for all $\phi \in C^{m,\alpha}$.

Before going to the proof of (ii) for $U_{[m]}$, $m = 1, 2, \ldots$, we study the operator $U_{[0]}$ defined by

$$U_{[0]}\phi(x) \equiv U\phi(x) - \int_0^1 \mathrm{d}x'\, \phi(x') \tag{44}$$

on the space $C_p^{m,\alpha}$ of periodic functions.

The spectral radius of $U_{[0]}$ is

$$R(U_{[0]}) = \frac{1}{\beta^{m+\alpha}}. \tag{45}$$

Proof of formula (45). We first observe that the spectral radius $R(U_{[0]})$ is bounded from below by $1/\beta^{m+\alpha}$, as for any z within the open disk $|z| < 1/\beta^{m+\alpha}$ the eigenfunction $\psi_z^s(x)$ is in the space $C_p^{m,\alpha}$. We show also that $R(U_{[0]}) \leq 1/\beta^{m+\alpha}$ by using the well-known formula for the spectral radius [36, Problem 88]

$$R(U_{[0]}) = \lim_{n\to\infty} \|U_{[0]}{}^n\|_{m,\alpha}{}^{1/n}. \tag{46}$$

Since we have

$$\|U_{[0]}^n\|_{m,\alpha} = \sup_{\phi\neq 0} \frac{\|U_{[0]}^n\phi\|_{m,\alpha}}{\|\phi\|_{m,\alpha}} = \sup_{\phi\neq 0} \frac{1}{\|\phi\|_{m,\alpha}} \left[\sum_{j=0}^{m} \|(U_{[0]}^n\phi)^{(j)}\|_\infty + \|(U_{[0]}^n\phi)^{(m)}\|_\alpha \right]$$

we should estimate the norms $\|(U_{[0]}^n\phi)^{(j)}\|_\infty$, $j = 0, 1, \ldots m$, and $\|(U_{[0]}^n\phi)^{(m)}\|_\alpha$.

In order to estimate the norms, we need the following lemma:

(iii) For all α-Hölder-continuous functions $g(x)$, $U_{[0]}g(x)$ is also α-Hölder-continuous (stability of the α-Hölder-continuous functions under $U_{[0]}$) and the following inequalities hold:

$$\left| \int_0^1 \mathrm{d}x\, \mathrm{e}^{-2\pi i n x} g(x) \right| \leq \frac{\|g\|_\alpha}{2^{\alpha+1}|n|^\alpha} \tag{47}$$

$$\|U_{[0]}g\|_\alpha \leq \frac{\|g\|_\alpha}{\beta^\alpha} \tag{48}$$

$$\|U_{[0]}g\|_\infty \leq \frac{\|g\|_\alpha}{\beta^\alpha}. \tag{49}$$

Proof of lemma (iii). (i) From the formula [41, p 24]:

$$\int_0^1 \mathrm{d}x\, \mathrm{e}^{-2\pi i n x} g(x) = \frac{1}{2} \int_0^1 \mathrm{d}x\, \mathrm{e}^{-2\pi i n x} \left\{ g(x) - g\left(x + \frac{1}{2n}\right) \right\} \tag{50}$$

and the α-Hölder continuity $|g(x)-g(y)| \le \|g\|_\alpha |x-y|^\alpha$, we obtain, using (47),

$$\left|\int_0^1 dx e^{-2\pi i n x} g(x)\right| \le \frac{1}{2}\int_0^1 dx \left|g(x) - g\left(x + \frac{1}{2n}\right)\right| \le \frac{\|g\|_\alpha}{2^{\alpha+1}|n|^\alpha}.$$

(ii) Inequality (48) as well as the stability of the α-Hölder-continuous functions follow from

$$\left|U_{[0]} g(x) - U_{[0]} g(y)\right| \le \frac{1}{\beta}\sum_{r=0}^{\beta-1}\left|g\left(\frac{x+r}{\beta}\right) - g\left(\frac{y+r}{\beta}\right)\right| \le \frac{\|g\|_\alpha}{\beta^\alpha}|x-y|^\alpha .$$

(iii) Since $\int_0^1 dx' g(x') = 1/\beta \sum_{r=0}^{\beta-1}\int_0^1 dy g(y/\beta + r/\beta)$, we have

$$\begin{aligned}|U_{[0]} g(x)| &\le \frac{1}{\beta}\sum_{r=0}^{\beta-1}\int_0^1 dy\left|g\left(\frac{x+r}{\beta}\right) - g\left(\frac{y+r}{\beta}\right)\right| \\ &\le \frac{1}{\beta}\sum_{r=0}^{\beta-1}\frac{\|g\|_\alpha}{\beta^\alpha}\int_0^1 dy|x-y|^\alpha \le \frac{\|g\|_\alpha}{\beta^\alpha}\end{aligned}$$

which implies $\|U_{[0]} g\|_\infty \le \|g\|_\alpha / \beta^\alpha$.

Estimation of $\|(U_{[0]}{}^n \phi)^{(m)}\|_\alpha$. Let $\phi(x) \in C_p^{(m,\alpha)}$; then $\int_0^1 dx \phi'(x) = \phi(1) - \phi(0) = 0$ and we have

$$\frac{d}{dx} U_{[0]}\phi(x) = \frac{1}{\beta} U \frac{d\phi}{dx}(x) = \frac{1}{\beta} U_{[0]}\frac{d\phi}{dx}(x) .$$

Similarly, as $\phi^{(\nu)}(1) = \phi^{(\nu)}(0)$ $(0 \le \nu \le m-1)$,

$$\frac{d^m}{dx^m} U_{[0]}{}^n \phi(x) = \frac{1}{\beta^{mn}} U_{[0]}{}^n \phi^{(m)}(x) . \tag{51}$$

Since $\phi^{(m)}(x)$ is α-Hölder-continuous, lemma (48) gives

$$\begin{aligned}\|(U_{[0]}{}^n \phi)^{(m)}\|_\alpha &= \frac{1}{\beta^{mn}}\|U_{[0]}{}^n \phi^{(m)}\|_\alpha \le \frac{1}{\beta^{mn}}\frac{1}{\beta^\alpha}\|U_{[0]}{}^{n-1}\phi^{(m)}\|_\alpha \\ &\le \cdots \le \left(\frac{1}{\beta^{m+\alpha}}\right)^n \|\phi^{(m)}\|_\alpha .\end{aligned} \tag{52}$$

Estimation of $\|(U_{[0]}{}^n \phi)^{(m)}\|_\infty$. Inequalities (48), (49) of lemma and (51) give

$$\begin{aligned}\|(U_{[0]}{}^n \phi)^{(m)}\|_\infty &= \frac{1}{\beta^{mn}}\|U_{[0]}{}^n \phi^{(m)}\|_\infty \le \frac{1}{\beta^{mn}}\frac{1}{\beta^\alpha}\|U_{[0]}{}^{n-1}\phi^{(m)}\|_\alpha \\ &\le \cdots \le \left(\frac{1}{\beta^{m+\alpha}}\right)^n \|\phi^{(m)}\|_\alpha .\end{aligned} \tag{53}$$

Estimation of $\|(U_{[0]}{}^n\phi)^{(\nu)}\|_\infty$ $(\nu = 0, 1, \ldots m-1)$. The function $\phi(x)$ can be expanded into a Fourier series

$$\phi(x) = \int_0^1 \mathrm{d}x'\phi(x') + \sum_{s\in Z_r}\sum_{k=0}^{\infty} c_k^s g_k^s(x) \tag{54}$$

where c_k^s is the Fourier coefficient

$$c_k^s = \int_0^1 \mathrm{d}x' g_k^{s*}(x')\phi(x') = \int_0^1 \mathrm{d}x' \exp(-2\pi \mathrm{i} s\beta^k x')\phi(x') \,. \tag{55}$$

Thus, from (3.4), we obtain

$$U_{[0]}{}^n\phi(x) = \sum_{s\in Z_r}\sum_{k=n}^{\infty} c_k^s g_{k-n}^s(x) = \sum_{s\in Z_r}\sum_{k=0}^{\infty} c_{k+n}^s g_k^s(x) \tag{56}$$

which leads to

$$|(U_{[0]}{}^n\phi)^{(\nu)}(x)| \le \sum_{s\in Z_r}\sum_{k=0}^{\infty}(2\pi|s|\beta^k)^\nu |c_{k+n}^s| \,. \tag{57}$$

As $\phi(x)$ is m-times continuously differentiable and $\phi^{(\nu)}(1) = \phi^{(\nu)}(0)$ $(0 \le \nu \le m-1)$, we obtain

$$\begin{aligned} |c_{k+n}^s| &= \left(\frac{1}{|2\pi \mathrm{i} s\beta^{k+n}|}\right)^m \left|\int_0^1 \mathrm{d}x' \exp(-2\pi \mathrm{i} s\beta^{k+n}x')\phi^{(m)}(x')\right| \\ &\le \frac{\|\phi^{(m)}\|_\alpha}{2^{\alpha+1}(2\pi)^m}\left(\frac{1}{|s|\beta^k}\right)^{m+\alpha}\left(\frac{1}{\beta^{m+\alpha}}\right)^n \end{aligned} \tag{58}$$

where the inequality (48) of lemma and the fact that $\phi^{(m)}$ is α-Hölder-continuous have been used. Thus, since $m + \alpha - \nu > 1$ for $0 \le \nu \le m-1$, we obtain

$$\begin{aligned} \|(U_{[0]}{}^n\phi)^{(\nu)}\|_\infty &\le \frac{\|\phi^{(m)}\|_\alpha}{2^{\alpha+1}(2\pi)^{m-\nu}}\left(\frac{1}{\beta^{m+\alpha}}\right)^n \sum_{s\in Z_r}\frac{1}{|s|^{m+\alpha-\nu}}\sum_{k=0}^{\infty}\left(\frac{1}{\beta^{m+\alpha-\nu}}\right)^k \\ &\le \frac{\|\phi^{(m)}\|_\alpha}{2^{\alpha}(2\pi)^{m-\nu}}\frac{\zeta(m+\alpha-\nu)}{1-\beta^{\nu-\alpha-m}}\left(\frac{1}{\beta^{m+\alpha}}\right)^n \end{aligned} \tag{59}$$

where $\zeta(\beta) \equiv \sum_{n=1}^{\infty} 1/n^\beta$ is Riemann's zeta function.

By combining (54), (55) and (59), we have

$$\|U_{[0]}{}^n\phi\|_{m,\alpha} = \sum_{\nu=0}^{m}\|(U_{[0]}{}^n\phi)^{(\nu)}\|_\infty + \|(U_{[0]}{}^n\phi)^{(m)}\|_\alpha \leqslant M\frac{\|\phi^{(m)}\|_\alpha}{\beta^{n(m+\alpha)}} \le M\frac{\|\phi\|_{m,\alpha}}{\beta^{n(m+\alpha)}} \tag{60}$$

where M is a positive constant depending only on m and α:

$$M = 2 + \sum_{\nu=0}^{m-1}\frac{1}{2^\alpha(2\pi)^{m-\nu}}\frac{\zeta(m+\alpha-\nu)}{1-\beta^{\nu-\alpha-m}} \,. \tag{61}$$

Thus we have

$$\|U_{[0]}^n\|_{m,\alpha} = \sup_{\phi\neq 0} \frac{\|U_{[0]}^n \phi\|_{m,\alpha}}{\|\phi\|_{m,\alpha}} \leq \frac{M}{\beta^{n(m+\alpha)}} .$$

From the previous estimations and formula (46) we obtain the desired upper bound for the spectral radius

$$R(U_{[0]}) = \lim_{n\to\infty} \|U_{[0]}{}^n\|_{m,\alpha}{}^{1/n} \leq \lim_{n\to\infty} M^{1/n} \frac{1}{\beta^{m+\alpha}} = \frac{1}{\beta^{m+\alpha}} .$$

We can now prove that the spectral radius of $U_{[m]}$ is also

$$R(U_{[m]}) = \frac{1}{\beta^{m+\alpha}} . \tag{62}$$

As in the case of $U_{[0]}$, the spectral radius $R(U_{[m]})$ is bounded by $1/\beta^{m+\alpha}$ from below. The argument is the same as before with the observation that

$$(U_{[m]} - U_{[0]})\psi_z^s(x) = -\sum_{n=1}^{m} \frac{\psi_z^{s(n-1)}(1) - \psi_z^{s(n-1)}(0)}{n!\,\beta^n} B_n(x) = 0 . \tag{63}$$

As in the case of $U_{[0]}$, we show that $R(U_{[m]}) \leq 1/\beta^{m+\alpha}$.

Estimation of $\|U_{[m]}{}^n\phi\|_{m,\alpha}$. Because the range of $U_{[m]}$ is $C_p^{m,\alpha}$ and $U_{[m]} = U_{[0]}$ on $C_p^{m\alpha}$, we can apply (60):

$$\|U_{[m]}{}^n\phi\|_{m,\alpha} = \|U_{[0]}{}^{n-1}U_{[m]}\phi\|_{m,\alpha} \leq M\beta^{m+\alpha} \frac{\|U_{[m]}\phi\|_{m,\alpha}}{\beta^{n(m+\alpha)}}$$

$$\leq M\beta^{m+\alpha}\|U_{[m]}\|_{m,\alpha} \frac{\|\phi\|_{m,\alpha}}{\beta^{n(m+\alpha)}} .$$

Therefore

$$R(U_{[m]}) = \lim_{n\to\infty} \|U_{[m]}{}^n\|_{m,\alpha}{}^{1/n} \leq \lim_{n\to\infty} (M\beta^{m+\alpha}\|U_{[m]}\|_{m,\alpha})^{1/n} \frac{1}{\beta^{m+\alpha}} = \frac{1}{\beta^{m+\alpha}} .$$

7. Concluding remarks

(1) The operator method (section 2) to construct the spectral decomposition is quite general. If exact solutions are not possible, the algorithm gives approximations to the eigenvalues and eigenfunctions. The method has been applied to other dynamical systems like the β-adic baker map, the Gauss map and the Friedrichs–Lee model [49].

(2) As the construction of the spectral decomposition is explicit, the choice of the test functions of the rigged Hilbert space is suggested by the formal result itself.

(3) The Koopman operator $U^\dagger$ being a unilateral shift [36,37] is to our knowledge the first example of operators, which do not have any kind of spectral

theorem in Hilbert space but possess a generalized eigenvector (30) and spectral decomposition (31). Previous work on generalized eigenfunction expansion was restricted to operators which admit a kind of spectral theorem in Hilbert space [50] like self-adjoint, unitary [38–40] and normal operators [51, 52].

(4) The term coherent states for the non-polynomial eigenfunctions of the Frobenius–Perron operator (section 5) is fully justifiable [53] in view of the generalizations [47] of the original quantum mechanical concepts. In fact, one can generalize the discussion of the Frobenius–Perron operators and introduce coherent states to any unilateral or bilateral shift operator [48].

(5) For expanding maps, Tangerman [13], Pollicott [18] and Ruelle [16, 17] have studied the dependence of the spectrum of the Frobenius–Perron operator on the smoothness of the domain. They showed that the essential spectral radius of the Frobenius–Perron operator decreases as the smoothness of the domain increases by a factor $\theta^{m+\alpha}$ determined by the smoothness $m + \alpha$ and expansion rates of the maps (θ being the inverse of the minimal expansion rate). The present results about the β-adic Renyi map also provide a concrete illustration of their theory. However, the explicit formulae for the eigenfunctions corresponding to the essential spectrum and the spectral decomposition involving only Pollicott–Ruelle resonances cannot be obtained within the present stage of development of the general theory by Tangerman [13], Pollicott [18] and Ruelle [16, 17].

(6) As we have seen in section 6, the essential spectral radius decreases as the smoothness of the functions in the domain increases. This phenomenon is easily understood from the smoothness of the non-polynomial eigenfunctions because, when the domain of the Frobenius–Perron operator U is restricted to the space of m-times continuously differentiable functions, the coherent eigenfunctions $\psi_z^s(x)$ corresponding to eigenvalues $1/\beta^m \leq |z| < 1$ are excluded. However, there exist infinitely differentiable linear combinations of the excluded eigenfunctions associated with isolated eigenvalues in the annulus $1/\beta^m \leq |z| < 1$. These isolated eigenvalues are just the Pollicott–Ruelle resonances. Saphir and Hasegawa [54] recently applied our coherent states (after our personal communication) in order to give another illustration of the dependence of the decay rates upon the smoothness of the test functions in the special case of the dyadic ($\beta = 2$) Renyi map.

(7) The admissible test function spaces $\mathcal{P}$ and $\mathcal{E}_c$, $c < 2\pi$ being at least analytic, exclude the Dirac delta functions. This means that the trajectories $\delta(y' - S^n y)$, $n = 0, 1, 2, \ldots$ are excluded from the domain of the spectral decomposition (22). Formula (22) can be used for probabilistic predictions using initial densities expandable in terms of admissible test functions only. This remark, however, goes beyond the prediction problem, as it also reflects the intrinsically probabilistic character of unstable dynamical systems.

The spectral decomposition (22) has, moreover, the property that the dynamical properties are reflected in the spectrum because the eigenvalues are the powers of the Lyapounov time.

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Appendix. The matrix elements of C_n and D_n

Because both U_1 and C_n are triangular matrices without diagonal parts, the equation (14*a*) for the matrix elements C_{mn} (16*a*) turns out to be

$$C_{mn} = \frac{1}{1/\beta^n - 1/\beta^m}\left[U_{mn} + \sum_{l=m+1}^{n-1} U_{ml}C_{ln}\right] \qquad n \geq m+2 \tag{A1a}$$

$$C_{n-1,n} = \frac{1}{1/\beta^n - 1/\beta^{n-1}}U_{n-1,n} = -\frac{1}{2}\frac{n!}{(n-1)!}\,. \tag{A1b}$$

Observing that $\beta^n m! U_{mn}/n!$ depends only on the difference $n-m$

$$\frac{m!}{n!}\beta^m\, U_{mn} = \frac{1}{(n-m)!}\frac{1}{\beta^{n-m+1}}\sum_{k=1}^{\beta-1} k^{n-m} \equiv f(n-m) \tag{A2}$$

one can rewrite (A1*a*) as

$$\frac{m!}{n!}C_{mn} = \frac{1}{\beta^{m-n}-1}\left[f(n-m) + \sum_{l=m+1}^{n-1} f(l-m)\frac{l!}{n!}C_{ln}\right]. \tag{A3}$$

From (A1*b*) and (A3), one find that $n!C_{mn}/m!$ depends only on the difference $n-m$ and thus we can set

$$\frac{m!}{n!}C_{mn} \equiv C(n-m)$$

which satisfies

$$\begin{aligned} C(n-m) &= \frac{1}{\beta^{m-n}-1}\left[f(n-m) + \sum_{l=1}^{n-m-1} C(n-m-l)f(l)\right] \\ &\quad n-m \geq 2 \\ C(1) &= -\frac{1}{2} = \frac{1}{1-\beta}f(1)\,. \end{aligned} \tag{A4}$$

As the second term of (A4) is a form of convolution, the sum equation (A4) can be solved through the generating function $\hat{C}(z)$ of $C(n)$:

$$\hat{C}(z) \equiv \sum_{n=1}^{\infty} z^n C(n)\,.$$

Indeed, multiplying $z^{n-m}(1-\beta^{n-m})$ by (A4) and summing it up with respect to $n-m$ from 1 to ∞, we obtain

$$\hat{C}(z/\beta) - \hat{C}(z) = \hat{f}(z) + \hat{C}(z)\hat{f}(z) \tag{A5}$$

with

$$\hat{f}(z) \equiv \sum_{n=1}^{\infty} z^n f(n) = \frac{1}{\beta}\sum_{k=1}^{\beta-1}\sum_{n=1}^{\infty}\frac{(kz/\beta)^n}{n!} = \frac{1}{\beta}\sum_{k=0}^{\beta-1}(e^{kz/\beta}-1) = \frac{e^z-1}{\beta(e^{z/\beta}-1)} - 1 .$$

Equation (A5) leads to the following functional equation for $h(z) \equiv (e^z-1)(1+\hat{C}(z))/z$:

$$h(z) = h(z/\beta) .$$

Thus, $h(z)$ is a 0th-order homogeneous function, i.e. constant:

$$h(z) = \frac{e^z-1}{z}[1+\hat{C}(z)] = h(0) = 1 .$$

Therefore,

$$1+\hat{C}(z) = 1+\sum_{n=1}^{\infty} z^n C(n) = \frac{z}{e^z-1} . \tag{A6}$$

As $z/(e^z-1)$ is the generating function of the Bernoulli numbers (17), we obtain

$$C(n) = \frac{B_n}{n!} .$$

This completes the derivation of (16*a*).

The equation (14*b*) for D_n for the matrix elements D_{nm} (16*b*) takes the form

$$\frac{n!}{m!}D_{nm} = \frac{1}{\beta^{m-n}-1}\left[\beta^{m-n}f(m-n) + \sum_{l=n+1}^{m-1}\beta^{m-l}f(m-l)\frac{n!}{l!}D_{nl}\right]$$

$$m \geq n+2 \tag{A7a}$$

$$\frac{n!}{(n+1)!}D_{n,n+1} = \frac{1}{\beta-1}\beta f(1) . \tag{A7b}$$

Thus, as in the case of C_n, $n!D_{nm}/m!$ depends only on the difference $m-n$:

$$\frac{n!}{m!}\langle\tilde{\varphi}_n|D_n|\varphi_m\rangle \equiv D(m-n)$$

which satisfies

$$D(m-n) = \frac{1}{\beta^{m-n}-1}\left[\beta^{m-n}f(m-n) + \sum_{l=1}^{m-n-1}\beta^{m-n-l}f(m-n-l)D(l)\right] . \tag{A8}$$

As in the case of C_n, (A8) can be solved through the generating function

$$\hat{D}(z) \equiv \sum_{n=1}^{\infty} z^n D(n) .$$

Multiplying $z^{m-n}(\beta^{m-n}-1)$ by (A8) and summing it up with respect to $m-n$, we obtain

$$\hat{D}(\beta z) - \hat{D}(z) = G(z) + G(z)\hat{D}(z) \tag{A9}$$

with

$$G(z) \equiv \sum_{n=1}^{\infty} z^n \beta^n f(n) = \frac{e^{\beta z}-1}{\beta(e^z-1)} - 1 .$$

Equation (A9) then leads to the following functional relation for $g(z) \equiv z(1+\hat{D}(z))/(e^z-1)$:

$$g(\beta z) = g(z) .$$

Thus, $g(z)$ is a 0th-order homogeneous function, i.e. constant:

$$g(z) = \frac{z[1+\hat{D}(z)]}{e^z-1} = g(0) = 1$$

which leads to

$$1 + \hat{D}(z) = 1 + \sum_{n=1}^{\infty} z^n D(n) = \frac{e^z - 1}{z} = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!} . \tag{A10}$$

Therefore, the non-vanishing matrix elements of D_n are given by (16*b*).

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